Finding the capacitance and inductance matrices for overhead lines

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1 Introduction

The current on overhead lines is ordinarily either oscillating, with a relatively low frequency, or direct. As such, the associated wavelength is very much longer than the typical distance between the lines, or the height of the lines above the ground. Consequently the electromagnetic field behaves very much as it would for a static problem, and the radiative terms can be ignored.

2 Line voltages

It is normal to specify the line voltage, that is the root mean square of the electro-motive force between any two of the energised lines. The voltage amplitude for each such line follows from that. However the electromagnetic field is much more easily obtained from a knowledge of the surface charges. These charges can reside on the surfaces of the lines, and on the ground plane. Thus a key for solving the electromagnetic field problem is to determine the surface charges that would give rise to the specified voltages.

3 Capacitance matrix

Where problems involving multiple surfaces are concerned, it is normal to replace the scalar capacitance with a capacitance matrix. Here individual rows correspond with individual surfaces. The same thing is true for columns. They correspond in the same way with the same surfaces. The element in a given row and column then represents the capacitance between the two corresponding surfaces. The surface charges can be obtained by premultiplying the voltage vector by the capacitance matrix. The question of determining the surface charges that would give rise to the specified voltages is therefore equivalent to determination of the capacitance matrix.

3.1 Self capacitance

If the capacitance can be represented by a matrix, then the question of what is meant by the diagonal terms of the capacitance matrix naturally arises. These are referred to as the self capacitances. Define the background potential at a surface to mean that part of the scalar potential which is due entirely to the charges on the other surfaces. Then the charge on a surface is equal to the self capacitance of that surface times the difference between its scalar potential and the background potential.

4 The Green's function

In two dimensional problems, determining the scalar potential from the charge density is relatively straightforward. The Green's function is essentially equal to the logarithm of the distance from the charge.

$$G_\rho\left(\mathbf{r_1}, \mathbf{r_2}\right) = -\frac{1}{2\pi\epsilon_0} ln \left|\mathbf{r_1} - \mathbf{r_2}\right| \tag{1}$$

The scalar potential at a point is then determined from the sum of all the charges multiplied by their respective values of the Green's function.

5 Conductor radius

Properly the Green's function corresponds with charge density. The potential is obtained by multiplying the Green's function by the charge density, and integrating this over all volumes containing non-zero charge densities. In the overhead line problem, it is normal to replace charge density by charge per unit length on each line. Provided that the lines are well separated, and that the potential is evaluated at a place not on or within a line, this produces a satisfactory result. However some care is required to evaluate the potential on a line due to charges on that line. The charges exist entirely on the surface of the conductor. In that case the potential due to those charges becomes steadily greater (assuming that the total charge on the surface is positive) towards the surface of the conductor, reaching a maximum value at that surface. Inside the conductor it does not change, for if it did, then there would be an electric field inside the conductor and the charges would move to negate it. The top of the potential hill is flat. The equivalent formula for the Green's function due to the surface charges is

$$G_q\left(\mathbf{r_1}, \mathbf{r_2}\right) = -\frac{1}{2\pi\epsilon_0} ln \left(max\left(\left|\mathbf{r_1} - \mathbf{r_2}\right|, a\right)\right) \tag{2}$$

where a represents the conductor radius.

5.1 Conductor bundles

It is quite common for overhead lines to consist of bundles of conductors. Each bundle is held together with spacing frames, and these frames are located at regular intervals along the route. A bundle typically contains two or four conductors. The frames are ordinarily small compared with the line separation. However the potential on any conductor in a bundle is influenced by the charges on the other conductors in the bundle. The charge divides equally between the conductors. The potential on any conductor in a twin conductor bundle is then (ignoring all other bundles)

$$\phi = -\frac{1}{2\pi\epsilon_0} ln\left(\sqrt{2rR}\right) \tag{3}$$

where r is the conductor radius and R is the radius of the bundle circle (which contains all the centres of the conductors in that bundle). Hence the equivalent radius of a single conductor is given by

$$a_2 = \sqrt{2rR} \tag{4}$$

This expression should then be substituted for a in the expression for the Green's function G_q associated with charges on single conductors.

The corresponding potential for a quadruple conductor bundle is

$$\phi = -\frac{1}{2\pi\epsilon_0} ln\left(\left(4rR^3\right)^{\frac{1}{4}}\right) \tag{5}$$

In this case the equivalent radius of a single conductor is given by

$$a_4 = \left(4rR^3\right)^{\frac{1}{4}} \tag{6}$$

6 The ground plane

For the overhead line problem, some care is required in specifying the position of the charges. There are charges on the overhead lines, and induced charges on the ground plane. The electric field lines should be perpendicular to the ground at the ground plane. That is, they should all be vertical at the ground plane.

6.1 The method of images

The distribution of induced charges satisfies that condition. A way of avoiding the problem of calculating the ground plane charge distribution is provided by the method of images. Consider a vertical plane which is normal to the path taken by the overhead lines. The ground plane intersects this plane in a line which may be regarded as the x axis. The y axis then extends vertically upwards from the ground plane, with the origin immediately below the path of the lines (the exact location is irrelevant). Each line (which is typically a metallic conductor) intersects this vertical plane in a small circle (or some other figure, depending on the conductor shape).

In the method of images, the charges that are induced on the ground plane are removed. Image lines are then placed below the ground plane, exactly the same distance below the x axis as the actual lines are above it. Charges are placed on the image lines. They are constrained to be equal and opposite to the corresponding charges on the actual lines. Solving the field problem for this geometry automatically satisfies the required boundary condition that the field lines are vertical on the x-axis.

7 Finding the scalar potential

The potential in the image problem is obtained by multiplying the Green's function by the charges, and adding them, as usual. However each conductor has an image conductor with equal and opposite charge. Thus the sum of the contributions from an actual line and its corresponding image line is equal to the charge on the actual line times the Green's function for the actual line minus the Green's function for the image line. Thus it is only necessary to sum over actual lines with this difference of Green's functions replacing the original Green's function.

$$G_I\left(\mathbf{r_1}, \mathbf{r_2}\right) = G_q\left(\mathbf{r_1}, \mathbf{r_2}\right) - G_q\left(\mathbf{r_1}, \mathbf{r_2} - 2\,\mathbf{r_2}\cdot\mathbf{e_y}\,\mathbf{e_y}\right) \tag{7}$$

The resultant potential is zero everywhere on the x axis, which corresponds with the ground plane. The potential at infinity is zero in all directions moving away from the origin because the Green's functions from each conductor and its corresponding image conductor tend to cancel.

8 Determination of the capacitance matrix

The scalar potential on any conductor can then be determined by adding the contributions due to the charges on all conductors (including that conductor itself). The relationship is linear and can be expressed in matrix form.

$$\mathbf{V} = [G_I]\,\mathbf{q} \tag{8}$$

The vector of scalar potentials, of which each element is just the scalar potential on a particular conductor, is equal to a square coefficient matrix times the vector of charges. The capacitance matrix is then determined by inverting that coefficient matrix.

$$\mathbf{C} = [G_I]^{-1} \tag{9}$$

9 Electric currents and the inductance matrix

Of course, charges are not the only source terms in the electromagnetic field equations. The other important source terms are electric currents. They give rise to another potential, called the magnetic vector potential. The magnetic field can be deduced from the vector potential. The magnetic vector potential depends on its source terms in basically the same way that the scalar potential depends on its source terms (for details of this dependence, see the Feynman Lectures on Physics, Volume 2, Table 15-1). The only difference is that in the latter case there is a division of the currents by a factor of the square of the velocity of light. If the problem is two dimensional, and the currents are in the same places as the charges, then the way that magnetic vector potential depends on currents is essentially the same as the way that scalar potential depends on charges. This dependence can be expressed in terms of the inductance matrix.

$$\mathbf{L} = c^{-2}\,[G_I] \tag{10}$$

Since the capacitance matrix is obtained by inverting this dependence, it follows that the capacitance matrix and inductance matrix are essentially inverse. The product of the inductance matrix and the capacitance matrix, multiplied by the square of the speed of light, is equal to the identity matrix.

$$c^2\mathbf{LC} = \mathbf{I} \tag{11}$$